

# Phase Space Distributions from Three-Port Couplers

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## Abstract

A wide class of phase space distributions of a single mode radiation field is shown to be directly accessible to measurement by linear symmetric three-port optical couplers.

In a classical description of the harmonic oscillator the measurement of some variable can be performed, at least in principle, with arbitrary precision. In particular, a pair of conjugated variables can jointly be measured, thus providing a unique correspondence between states of the oscillator and points in the phase space of complex amplitudes. This is no longer true in quantum mechanics, to which classical phase space description cannot be transferred [1]. Two main features have to be considered: i) the uncertainty principle which prevents precise identification of points in phase space and ii) the commutation relations which lead to ordering rules for representing physical variables in terms of operators [2]. Nevertheless, a phase space description of the harmonic oscillator has been very fruitful in quantum optics and different distribution functions have been introduced to describe various dynamical processes. Some examples are laser theory, the study of the phase properties of light [3] and the quantum state measurement problem [4, 5].

An important class of quantum phase space distributions are the so-called generalized Wigner functions [2]

$$W_s(\alpha, \bar{\alpha}) = \int \frac{d^2\lambda}{\pi} \chi_s(\lambda, \bar{\lambda}) e^{\bar{\lambda}\alpha - \lambda\bar{\alpha}}, \quad (1)$$

where

$$\chi_s(\lambda, \bar{\lambda}) = \text{Tr} \left\{ \hat{\rho} e^{\lambda a^\dagger - \bar{\lambda} a + \frac{1}{2}s|\lambda|^2} \right\} \quad (2)$$

is the  $s$ -ordered characteristic function of the field  $\hat{\rho}$  and is related to different ordering of the boson operators  $a$ ,  $a^\dagger$  since the statistical average  $\langle \bar{\alpha}^k \alpha^l \rangle_s$  – employing the quasi-probability distribution  $W_s(\alpha, \bar{\alpha})$  – provides the quantum expectation value of the  $s$ -ordered operator product  $\{a^{\dagger k} a^l\}_s$ .

For  $s = 1$  we get Glauber's  $\mathcal{P}$ -function [6], for  $s = 0$ , the distribution originally proposed by Wigner [1] and for  $s = -1$  the Husimi  $Q$ -function [7] which are related to normal, symmetric and antinormal ordering of the boson operators respectively. Such generalized Wigner functions can be sampled experimentally only for parameter  $s \leq -1$ , since for greater values of  $s$  negative or singular probabilities occur for some quantum states.

A more general class of phase distributions has been introduced in quantum optics [8]. These are defined as a convolution of the Wigner function of the signal  $W_{0|S}$  with that of the probe  $W_{0|P}$  and serve as a description of the signal in terms of the probe:

$$K_{SP}(\alpha, \bar{\alpha}) = \int_{\mathbf{C}} \frac{d^2\beta}{\pi^2} W_{0|S}(\alpha + \beta, \bar{\alpha} + \bar{\beta}) W_{0|P}(\beta, \bar{\beta}) . \quad (3)$$

After some calculation this can be shown to be equivalent to

$$K_{SP}(\alpha, \bar{\alpha}) = \frac{1}{\pi} \text{Tr} \left\{ \hat{\rho}_S \hat{D}(\alpha) \hat{\rho}_P \hat{D}^\dagger(\alpha) \right\} , \quad (4)$$

which represents  $K_{SP}$  in terms of the overlap between the signal field  $\hat{\rho}_S$  and the  $\alpha$ -displaced probe field  $\hat{\rho}_P$ ;  $\hat{D}(\alpha) = \exp\{\alpha a^\dagger - \bar{\alpha} a\}$  is the displacement operator. Phase space density as in Eq. (3) have been introduced to account for the effect of the measuring apparatus in a joint measurement of conjugated variables [8]. More recently, they also have been used in entropic descriptions of quantum mechanical states [9].

In this paper we consider *triple coupler homodyning* of a single mode radiation field, that is, the use of a linear three-port coupler to couple the signal beam with a strong local oscillator and also with a second probe field. Upon using vacuum as the probe mode input this leads to the measurement of Husimi's  $Q$ -function, a result which is a by-product of the present approach and is somewhat interesting since the measurement of the  $Q$ -function is commonly associated with an  $4 \times 4$  - port device [3, 10] (eight-port homodyne [11]) although we see here a  $3 \times 3$  - port is enough. For a general probe field we will find that triple coupler homodyning allows access to all possible phase space density  $K_{SP}$ , provided the corresponding field  $\hat{\rho}_P$  can be generated. The use of a lower number of modes results in an easier and less noisy experimental implementation.

A linear, symmetric three-port optical coupler is a generalization of the customary lossless symmetric beam splitter. The three input modes  $a_i$ ,  $i = 1, 2, 3$  are combined to form 3 output modes  $b_j$ ,  $j = 1, 2, 3$ . In analogy to lossless beam splitters which are described by unitary  $2 \times 2$  matrices [12], any lossless symmetric triple coupler is characterized by a unitary  $3 \times 3$  matrix [13, 14] of the form

$$\mathbf{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \exp\{i\frac{2\pi}{3}\} & \exp\{-i\frac{2\pi}{3}\} \\ 1 & \exp\{-i\frac{2\pi}{3}\} & \exp\{i\frac{2\pi}{3}\} \end{pmatrix} , \quad (5)$$

where each matrix element  $T_{ij}$  represents the transmission amplitude from the  $i$ -th input port to the  $j$ -th output port, that is  $b_j = \sum_{k=1}^3 T_{jk} a_k$ .

Such devices have already been implemented in single-mode optical fiber technology and commercial triple coupler have been available for some time [15]. It is known [16] that any unitary  $M$ -dimensional matrix can be factorized into a sequence of 2-dimensional transformation. Thus any triple coupler can be implemented by discrete optical components using 50:50 beam splitters and phase shifters only [13], which is schematically displayed in Fig. 1 together with our notation for general linear three-port couplers. Experimental realizations of triple couplers has been reported for both cases, the passive elements case and the optical fiber one [13, 14].

Let us now consider the measurement scheme of Fig. 2. The three input modes are mixed by a triple coupler and the resulting output modes are subsequently surveyed by three identical photodetector. The measured photocurrents are proportional to  $\hat{I}_n$ ,  $n = 1, 2, 3$  given by

$$\hat{I}_n = b_n^\dagger b_n = \frac{1}{3} \sum_{k,l=1}^3 \exp \{i\theta_n(l-k)\} a_k^\dagger a_l, \quad \theta_n = \frac{2\pi}{3}(n-1). \quad (6)$$

After photodetection a Fourier transform (FT) on the photocurrents is performed

$$\hat{\mathcal{I}}_s \equiv \text{FT}(\hat{I}_1, \hat{I}_2, \hat{I}_3) = \frac{1}{\sqrt{3}} \sum_{n=1}^3 \hat{I}_n \exp \{-i\theta_n(s-1)\}, \quad s = 1, 2, 3. \quad (7)$$

This procedure is a straightforward generalization of the customary two-mode balanced homodyning technique. By means of the identity

$$\delta_3(s-1) = \frac{1}{3} \sum_{n=1}^3 \exp \left\{ i \frac{2\pi}{3} n(s-1) \right\}, \quad (8)$$

for the periodic (modulus 3) Kronecker delta  $\delta_3$ , we obtain our final expressions for the Fourier transformed photocurrents

$$\hat{\mathcal{I}}_1 = \frac{1}{\sqrt{3}} \left\{ a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \right\}, \quad (9)$$

$$\hat{\mathcal{I}}_2 = \frac{1}{\sqrt{3}} \left\{ a_1^\dagger a_2 + a_2^\dagger a_3 + a_3^\dagger a_1 \right\}, \quad (10)$$

$$\text{and } \hat{\mathcal{I}}_3 = \frac{1}{\sqrt{3}} \left\{ a_1^\dagger a_3 + a_2^\dagger a_1 + a_3^\dagger a_2 \right\}. \quad (11)$$

$\hat{\mathcal{I}}_1$  gives no relevant information as it is insensitive to the phase of the signal field, whereas  $\hat{\mathcal{I}}_2$  and  $\hat{\mathcal{I}}_3$  are hermitian conjugates of each other and contain the relevant information in their real and imaginary part.

In the following let us assume  $a_1$  is the signal mode and  $a_2$  is fed by a highly excited coherent state  $|z\rangle$  representing the local oscillator. Since the local oscillator serves as the reference phase ( $\varphi = 0$ ), we have  $|z\rangle = ||z|\rangle$ . For large  $z$  the output photocurrents are intense enough to be easily detected and can be combined to give the reduced photocurrents

$$\begin{aligned}\hat{\mathcal{Y}}_1 &= \sqrt{3} \frac{\hat{\mathcal{I}}_2 + \hat{\mathcal{I}}_3}{2|z|} = \hat{a}_1(0) + \hat{a}_3(0) + O\left[\frac{1}{|z|}\right] \\ \hat{\mathcal{Y}}_2 &= \sqrt{3} \frac{\hat{\mathcal{I}}_2 - \hat{\mathcal{I}}_3}{2i|z|} = \hat{a}_1(-\pi/2) - \hat{a}_3(-\pi/2) + O\left[\frac{1}{|z|}\right],\end{aligned}\quad (12)$$

which we refer to as the *triple homodyne photocurrents*. In Eq. (12)  $\hat{a}(\varphi) = 1/2(a^\dagger e^{i\varphi} + a e^{-i\varphi})$  denotes a quadrature operator of the field.

Each experimental outcome from a tritter homodyne detector is a pair  $(y_1, y_2)$  of real numbers from the joint measurements of  $\hat{\mathcal{Y}}_1$  and  $\hat{\mathcal{Y}}_2$ . The corresponding two-dimensional probability distribution of such outcomes is given by

$$P(y_1, y_2) = \int_{\mathbf{R}} \frac{d\lambda_1}{2\pi} \int_{\mathbf{R}} \frac{d\lambda_2}{2\pi} e^{-i\lambda_1 y_1 - i\lambda_2 y_2} \Xi(\lambda_1, \lambda_2), \quad (13)$$

which is a double Fourier transform of the characteristic function  $\Xi(\lambda_1, \lambda_2)$  of the device, namely

$$\Xi(\lambda_1, \lambda_2) = \text{Tr} \left\{ (\hat{\rho}_1 \otimes \hat{\rho}_3) e^{i\lambda_1 \hat{\mathcal{Y}}_1 + i\lambda_2 \hat{\mathcal{Y}}_2} \right\}. \quad (14)$$

Inserting Eq. (12) in Eq. (14), the characteristic function can be rewritten in terms of the complex variable  $\gamma = (\lambda_2 + i\lambda_1)/2$  yielding

$$\begin{aligned}\Xi(\gamma, \bar{\gamma}) &= \text{Tr} \left\{ (\hat{\rho}_1 \otimes \hat{\rho}_3) e^{\gamma a_1^\dagger - \bar{\gamma} a_1} e^{-\gamma a_3^\dagger + \bar{\gamma} a_3} \right\} \\ &= \chi_{0|1}(\gamma, \bar{\gamma}) \chi_{0|3}(-\gamma, -\bar{\gamma}).\end{aligned}\quad (15)$$

It factorizes into two characteristic functions  $\chi_0$ , see Eq. (2), one for each mode. This and Eq. (13) – changed to variables  $\gamma$  – gives  $P$  as an integral over the complex plane

$$P(y_1, y_2) = \int_{\mathbf{C}} \frac{d^2\gamma}{\pi^2} e^{\bar{\gamma}(y_1 - iy_2) - \gamma(y_1 + iy_2)} \chi_{0|1}(\gamma, \bar{\gamma}) \chi_{0|3}(-\gamma, -\bar{\gamma}). \quad (16)$$

We find that  $P$  is a convolution integral reminiscent of  $K_{SP}$  in Eq. (3), together with Eq. (1) this leads to our *main result*:

$$P(y_1, y_2) = K_{13}(y_1 - iy_2, y_1 + iy_2). \quad (17)$$

It shows that the outcome probability distribution from a triple coupler homodyne detection gives the desired phase space density of Eqs. (3) and (4), the

mode  $a_3$  represents the probe mode  $P$  whereas  $a_1$  is the signal mode  $S$ . When the probe is left in the vacuum state, Eq. (15) reads

$$\begin{aligned}\Xi(\gamma, \bar{\gamma}) &= \text{Tr} \left\{ \hat{\rho}_1 \otimes |0\rangle\langle 0| e^{\gamma a_1^\dagger - \bar{\gamma} a_1} e^{-\gamma a_3^\dagger + \bar{\gamma} a_3} \right\} = \\ &= \chi_{0|1}(\gamma, \bar{\gamma}) e^{-\frac{1}{2}|\gamma|^2} = \chi_{-1|1}(\gamma, \bar{\gamma}),\end{aligned}\quad (18)$$

namely, the triple coupler homodyning device becomes a Husimi  $Q$ -function detector. Note that customary eight-port homodyne thus mimics this scheme with one more input and output.

We end the paper with Fig. 3, where we illustrate some examples of phase space ityropensities accessible by triple coupler homodyne detection. A coherent state of real amplitude  $\beta = 1$  is chosen for the signal: Feeding vacuum into the probe mode  $a_3$  the Husimi  $Q$ -function  $K_{|\beta\rangle|0\rangle}(\alpha, \bar{\alpha}) = Q(\alpha, \bar{\alpha}) = 1/\pi \exp\{-|\alpha - \beta|^2\}$  is obtained, if a squeezed vacuum  $|0, r\rangle$  is used instead, the distribution  $K_{|\beta\rangle|0, r\rangle}(\alpha, \bar{\alpha}) = 1/\pi |\langle \alpha, r | \beta \rangle|^2$  results, and a number state  $|n\rangle$  gives the distribution  $K_{|\beta\rangle|n\rangle}(\alpha, \bar{\alpha}) = 1/\pi |\langle n | \hat{D}^\dagger(\alpha) | \beta \rangle|^2$ .

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## References

- [1] E. P. Wigner, Phys. Rev. **40**, 749 (1932).
- [2] K. E. Cahill, R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
- [3] G. M. D’Ariano, M. G. A. Paris, Phys. Rev. **A49**, 3022 (1994); R. Tanaś, A. Miranowicz, Ts. Gantsog, Progress in Optics **35**, 355 (1996); B. K. Murzakhmetov, A. V. Chizhov, Zh. Fiz. Elem. Chast. i Atom. Yadra (1996), in press.
- [4] K. Vogel, H. Risken, Phys. Rev. **A40**, 2847 (1989).
- [5] G. M. D’Ariano, C. Macchiavello, M. G. A. Paris, Phys. Rev. **A50** 4298 (1994); G. M. D’Ariano, U. Leonhardt, H. Paul, Phys. Rev. **A52**, R1801, (1995); M. G. A. Paris, Phys. Rev. **A53**, 2658 (1996).
- [6] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963); E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
- [7] K. Husimi, Proc. Phys. Mat. Soc. Jpn. **22**, 264 (1940); Y. Kano, J. Math. Phys. **6**, 1913 (1965).

- [8] K. Wodkiewicz, Phys. Rev. Lett. **52**, 1064 (1984); Phys. Lett. **A115**, 304 (1986); Phys. Lett. **A129**, 1 (1988).
- [9] V. Buzek, C. H. Keitel, P. L. Knight, Phys. Rev. **A51**, 2575 (1995); Phys. Rev. **A51**, 2594 (1995).
- [10] U. Leonhardt, H. Paul, Phys. Rev. **47**, 2460 (1993); M. Freiberger, W. Vogel, W. Schleich, Phys. Lett. **176A**, 41 (1993).
- [11] N.G.Walker, J.E.Carrol, Opt. Quantum Electr. **18**, 355 (1986); N. G. Walker, J. Mod. Opt. **34**, 16 (1987); Y. Lay, H. A. Haus, Quantum Opt. **1**, 99 (1989); J. W. Noh, A. Fougères, L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992); Phys. Rev. A **46**, 2840 (1992).
- [12] R. A. Campos, B. E. A. Saleh, M. C. Teich, Phys. Rev **A42**, 4127 (1990).
- [13] K. Mattle, M. Michler, H. Weinfurter, A. Zeilinger and M. Zukowski, Appl. Phys. **B60**, S111 (1995).
- [14] G. Weihs, M. Reck, H. Weinfurter and A. Zeilinger, Opt. Lett. **21**, 302 (1996).
- [15] S. K. Sheem, J. Appl. Phys. **52**, 3865 (1981).
- [16] M. Reck, A. Zeilinger, H. J. Bernstein, P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

Figure 1: Schematic diagram of a triple coupler and its realization in terms of 50:50 beam splitters (BS) and phase shifters ' $\varphi$ '. In order to obtain a symmetric coupler the following values has to be chosen:  $\varphi_1 = \arccos(1/3)$  and  $\varphi_2 = \varphi_1/2$ .

Figure 2: Outline of triple coupler homodyne detectors: The hexagonal box symbolizes the electronically performed Fourier transform (FT).

Figure 3: Examples of phase space densities measurable by the triple coupler homodyne detection. For a coherent state of amplitude  $\beta = 1$  we find (a) the Husimi  $Q$ -function obtained if the probe mode  $a_3$  is the vacuum, (b) the distribution which results if the probe is a squeezed vacuum  $|0, r\rangle$  with the mean photon number  $\sinh^2 r = 1$ , and (c) the distribution which results if the probe is a number state  $|n\rangle$  with  $n = 1$ .

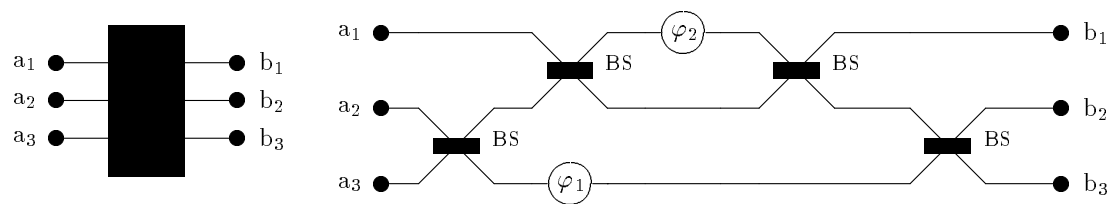


Fig. 1  
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*DA2563 Phase Space Distributions from ...*



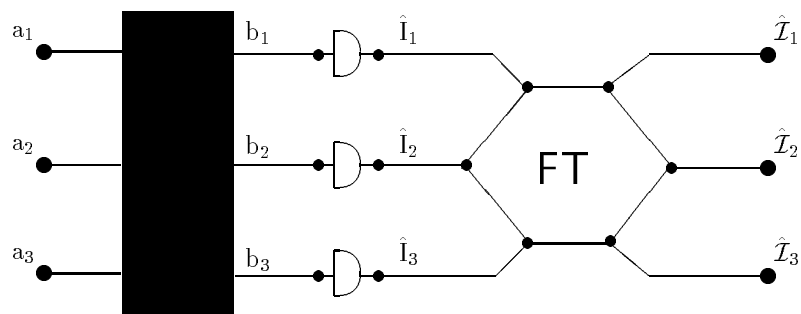
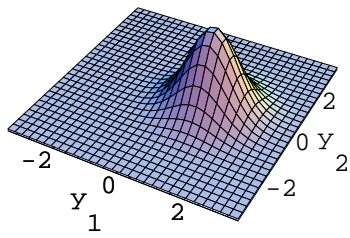


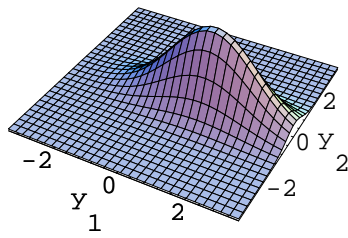
Fig. 2  
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Paris, Chizhov and Steuernagel Fig.3 DA2563 Phase Space Dis...

(a)



(b)



(c)

